

Supergravity dual of c -extremization

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Abstract

Recently a general principle, called c -extremization, which determines the exact R-symmetry of two-dimensional SCFTs with $\mathcal{N} = (0, 2)$ supersymmetry was identified. In this note we show that the supergravity dual corresponds to the extremization of the T -tensor of $\mathcal{N} = 2$ gauged supergravity in three-dimensions. To support this claim, we demonstrate that the expected central charge of SCFTs arising from twisted compactifications of four-dimensional $\mathcal{N} = 4$ SYM on Riemann surfaces, whose gravity dual is a reduction of five-dimensional $U(1)^3$ gauged supergravity, is recovered in the three-dimensional framework.

1. Introduction

c -extremization [1, 2] for two-dimensional superconformal field theories (SCFTs) shares much in common with its higher-dimensional counterpart, a -maximization [3]. Recall that for theories flowing to IR SCFTs, the R-symmetry is not unique and can mix with flavour symmetries leading to an ambiguity over the exact superconformal R-symmetry at the fixed point. The respective principles address this issue through the identification of a process in which one extremises a polynomial constructed from 't Hooft anomalies which are themselves recognised as invariants of RG flows.

Through the AdS/CFT correspondence [4, 5, 6], we can understand these SCFTs at large N in terms of supersymmetric anti-de Sitter space-times where, in the case of string/M-theory backgrounds, the R-symmetry $U(1)_R$ may be a particular linear combination of the $U(1)$ isometries of the internal geometry. For Sasaki-Einstein manifolds, a purely geometric extremisation principle [7, 8] explains how the Reeb vector dual to $U(1)_R$ can be determined by minimising the volume of the internal manifold. This process may be regarded as equivalent to a -maximization [9, 10].

More generally, a -maximization has an interpretation in terms of the minimisation of the Killing prepotential of $\mathcal{N} = 2$ gauged supergravity in five dimensions [11]. Recently, this procedure has been applied to a dimensional reduction [12] of a family of supergravity solutions [13, 14] corresponding to M5-branes wrapped on Riemann surfaces which includes the earlier examples of [15]¹. The geometric dual of a -maximization for the general class of geometries in [17] remains an open problem.

Recent developments beg the question what is the AdS/CFT dual description for c -extremization. To address this problem, we tailor the arguments of [11] to the language of three-dimensional gauged supergravity and, in the so-called T -tensor of $\mathcal{N} = 2$ gauged supergravity, we identify a function that, when extremised, produces the central charge. As we will discuss in the next section, for $\mathcal{N} = 2$ gauged supergravity, only the T -tensor T and the holomorphic superpotential W appear in the scalar potential. When the R-symmetry is gauged, consistency requires that $W = 0$ [18], so the potential depends only on T . Therefore, for $SO(2)_R \sim U(1)_R$ gauged supergravities with AdS_3 vacua, the extremisation of T naturally leads to the minimum of the potential.

2. Review of c -extremization

In a non-conformal $\mathcal{N} = (0, 2)$ supersymmetric theory with $U(1)_R$ R-symmetry, the R-symmetry is not uniquely defined and mixing of $U(1)_R$ with the other Abelian flavour symmetries is permitted. At a conformal fixed point this changes, and an exact superconformal R-symmetry is picked out. To identify this exact R-symmetry at the superconformal fixed point, [1, 2] introduced a “trial R-current”

$$\Omega_\mu^{\text{tr}}(t) = J_\mu^r + \sum_{M(\neq r)} t_M J_\mu^M, \quad (1)$$

where J_μ^r is a choice of the R-symmetry current and J_μ^M ($M \neq r$) are Abelian flavour symmetry currents. From $\Omega_\mu^{\text{tr}}(t)$ one constructs a quadratic function $c_R^{\text{tr}}(t)$ which is proportional to the 't Hooft anomaly of $\Omega_\mu^{\text{tr}}(t)$:

$$c_R^{\text{tr}}(t) = 3 \left(k^{rr} + 2 \sum_{M(\neq r)} t_M k^{rM} + \sum_{M, N(\neq r)} t_M t_N k^{MN} \right), \quad (2)$$

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¹The $\mathcal{N} = 1$ SCFT duals of [15] are discussed in [16].

where k^{MN} are the 't Hooft anomaly coefficients. Recall that these anomalies arise in the context of theories with $U(1)^P$ global symmetry when the theory is coupled to non-dynamical vector fields $A_\mu^M, M = 1, \dots, P$, in a curved background with metric $g_{\mu\nu}$. The anomalous violations of current conservation are then given by

$$\nabla^\mu J_\mu^M = \sum_N \frac{k^{MN}}{8\pi} F_{\mu\nu}^N \epsilon^{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = \frac{k}{96\pi} g^{\nu\alpha} \epsilon^{\mu\rho} \partial_\mu \partial_\beta \Gamma_{\alpha\rho}^\beta,$$

where $F^M = dA^M$, $T_{\mu\nu}$ is the stress tensor and $\Gamma_{\alpha\rho}^\beta$ is the Levi-Civita connection for $g_{\mu\nu}$.

The trial c -function (2) can be motivated from a study of the $\mathcal{N} = 2$ superconformal algebra [1, 2]. In particular, for supercharges \mathcal{Q} with R-charge 1, the algebra fixes a relation between the central charge c_R and the R-symmetry anomaly $c_R = 3k^{rr}$. In addition, it can be shown in a renormalization scheme where all currents are primary fields that there are no mixed anomalies between the superconformal R-current and flavour currents. This imposes the constraint $k^{rM} = 0, \forall M \neq r$, and leads to the extremality condition

$$\frac{\partial c_R^{\text{tr}}}{\partial t^M}(t_0) = 0, \quad \forall M \neq r. \quad (3)$$

Since $c_R^{\text{tr}}(t)$ is quadratic, there is a unique solution.

3. $\mathcal{N} = 2$ Supergravity

Here, following the notation of [18], we present a succinct review of $\mathcal{N} = 2$ gauged supergravity in three dimensions. The field content comprises scalar fields ϕ^i , spinor fields χ^i , both with $i = 1, \dots, d$, a dreibein e_μ^a , the spin-connection ω_μ^{ab} and two gravitini $\psi_\mu^I, I = 1, 2$, which transform under the R-symmetry group $SO(2)$.

The target space for the scalars is a Kähler manifold. As such, it is convenient to decompose the d real fields into $d/2$ complex ones and their corresponding complex conjugates, $\phi^i \rightarrow (\phi^i, \bar{\phi}^{\bar{i}})$. The Kähler manifold can then be locally written in terms of a metric $g_{i\bar{i}} = \partial_i \partial_{\bar{i}} \mathcal{K}$ where $\mathcal{K}(\phi, \bar{\phi})$ is the Kähler potential.

As explained in [18], a subgroup of isometries may be gauged through the introduction of an embedding tensor Θ_{MN} which defines the Killing vectors that generate the gauge group $X^i = g \Theta_{MN} \Lambda^N(x) X^{Ni}$, where g is the gauge coupling constant and $\Lambda^N(x)$ denotes the gauge group parameters. As is customary, the embedding tensor appears along with gauge fields A_μ^M in the definition of covariant derivative

$$\mathcal{D}_\mu \phi^i = \partial_\mu \phi^i + g \Theta_{MN} A_\mu^M X^{Ni}, \quad (4)$$

and also appears in the (Abelian) Chern-Simons (CS) term in the Lagrangian

$$\mathcal{L}_{CS} = \frac{1}{2} g \epsilon^{\mu\nu\rho} A_\mu^M \Theta_{MN} \partial_\nu A_\rho^N. \quad (5)$$

The embedding tensor also crops up in the T -tensor $T = 2\mathcal{V}^M \Theta_{MN} \mathcal{V}^N$ where \mathcal{V} is the moment map of the gauged isometries. We observe here that the T -tensor is quadratic in the moment maps, so structurally it bears some resemblance to the trial c -function (2).

Lastly, the scalar potential of the gauged theory may be expressed in terms of a real superpotential F :

$$V = -g^2 \left(8F^2 - 8g^{i\bar{i}} \partial_i F \partial_{\bar{i}} F \right), \quad (6)$$

where one can choose F to be one of the eigenvalues of the gravitino mass matrix $F = -T \pm e^{\mathcal{K}/2} |W|$, where W is the holomorphic superpotential satisfying $\partial_i \bar{W} = \partial_{\bar{i}} W = 0$. The potential tells us that, even in the absence of gauging, one can generate a cosmological constant with constant W . An alternative way to do this involves gauging the R-symmetry group, in which case T is a non-zero constant with $W = 0$. When the R-symmetry is gauged W must vanish since it transforms non-trivially under $SO(2)_R$.

3.1. Dual of c -extremization

Now that we have discussed the rudiments of $\mathcal{N} = 2$ gauged supergravity, we can recast the argument of [11] in terms of three-dimensional language. We start by noting that the embedding tensor Θ_{MN} encodes the CS terms and as observed in [2] these correspond to the 't Hooft anomalies k^{MN} . The exact relationship for wrapped D3-brane geometries we will introduce later.

Next, we remark that for two-dimensional superconformal theories, the corresponding AdS_3 dual geometry will preserve four supersymmetries. In particular, one can verify that the Killing spinor equations [18] are satisfied when $\partial_i T = 0$ since, with R-symmetry gauging, W has to be zero. Going further, from an analysis of the anti-commutator of the supercharges acting on the scalars, one can infer that the superconformal R-symmetry is

$$R = \tilde{s}^M Q_M = t \mathcal{V}^M Q_M, \quad (7)$$

where $Q_M, M = 1, \dots, P$, are charges corresponding to the currents J_μ^M and t is a constant of proportionality. As in [11], the gauge transformation for the gravitino [18]

$$\mathcal{D}_\mu \psi_\nu^I = \partial_\mu \psi_\nu^I + g \Theta_{MN} A_\mu^M \mathcal{V}^{NIJ} \psi_\nu^J \dots \quad (8)$$

allows us to use the fact that the gravitino has R-charge one to fix the constant of proportionality, i.e. $\tilde{s}^M \Theta_{MN} \mathcal{V}^N = 1$, leading to

$$\tilde{s}^M = 2 T^{-1} \mathcal{V}^M, \quad (9)$$

where T is the T -tensor we introduced earlier. We are now in a position to propose the supergravity trial c -function

$$c_R \propto \tilde{s}^N \Theta_{MN} \tilde{s}^M = 2 T^{-1}. \quad (10)$$

Observe that this trial function is extremised when $\partial_i T = 0$ which is precisely the condition for a supersymmetric

AdS_3 vacuum. Furthermore, for D3-branes wrapped on a Riemann surface Σ , we can infer the constant of proportionality from (3.15) of [2],

$$k^{MN} = \frac{\eta_\Sigma d_G}{2} \Theta_{MN}, \quad (11)$$

where d_G is the dimension of the gauge group G and η_Σ is related to the volume of the Riemann surface $\frac{1}{2\pi} \text{vol}_\Sigma = \eta_\Sigma$. We now recall that the trial c -function (2) is of the form $c_R \sim 3k^{MN} \sim \frac{3}{2} \eta_\Sigma d_G \Theta_{MN}$, where we have used (11). This suggests that the trial c -function from the supergravity perspective should be

$$c_R = \frac{3\eta_\Sigma d_G}{T}. \quad (12)$$

In the next section, we show that this formula recovers the expected central charge for the wrapped D3-brane geometries discussed in [1, 2].

4. AdS_3 vacua from D3-branes

In this section, to back up our claim, we revisit the initial example of c -extremization presented in [1] (later in [2]), but here recast it in the language of three-dimensional gauged supergravity. Our point of departure will be five-dimensional $U(1)^3$ gauged supergravity, which in turn may be embedded into type IIB supergravity in ten dimensions [19]. The action reads

$$\begin{aligned} e^{-1} \mathcal{L}_5 &= R - \frac{1}{2} \sum_i (\partial \varphi_i)^2 - \frac{1}{4} \sum_i X_i^{-2} F_{\rho\sigma}^i F^{i\rho\sigma} \\ &+ V_5 + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^1 F_{\rho\sigma}^2 A_\lambda^3, \end{aligned} \quad (13)$$

where e is the determinant of the vielbein, A^i denotes the gauge fields, V_5 labels the potential

$$V_5 = 4 \sum_i X_i^{-1}, \quad (14)$$

and for completeness we define the constrained scalars

$$X_1 = e^{-\frac{1}{2} \left(\frac{2}{\sqrt{6}} \varphi_1 + \sqrt{2} \varphi_2 \right)}, \quad X_2 = e^{-\frac{1}{2} \left(\frac{2}{\sqrt{6}} \varphi_1 - \sqrt{2} \varphi_2 \right)}, \quad (15)$$

with X_3 following from the constraint $X_1 X_2 X_3 = 1$. Observe also that for simplicity we have set the gauge coupling of the $U(1)^3$ theory to unity $g = 1$. This theory permits the following chain of further consistent truncations: $\{\varphi_2 = 0, F^1 = F^2\} \rightarrow U(1)^2$ gauging and $\{\varphi_1 = \varphi_2 = 0, F^1 = F^2 = F^3\} \rightarrow$ minimal gauged supergravity, where in the latter case the retained gauge field is the graviphoton.

To establish a connection to three dimensions, we adopt the following ansatz for five-dimensional space-time

$$ds_5^2 = e^{-4A} ds_3^2 + e^{2A} ds^2(\Sigma), \quad (16)$$

where A is a scalar warp factor and Σ is a Riemann surface with constant curvature $\kappa = -1, 0, 1$. In tandem, we take an appropriate ansatz for the field strengths

$$F^i = -a_i \text{vol}_\Sigma + G^i, \quad (17)$$

where closure of F^i implies that a_i are constants and that associated to each G^i we have gauge potential B^i , $G^i = dB^i$. In addition, we make the natural assumption that the scalars φ_i do not depend on the coordinates of the Riemann surface.

Plugging the ansatz into the five-dimensional equations of motion and reconstructing the Lagrangian, or alternatively performing the reduction at the level of the action, one finds a three-dimensional theory of the form

$$\begin{aligned} e^{-1} \mathcal{L}_3 &= R - 6(\partial A)^2 - \frac{1}{2} \sum_i (\partial \varphi_i)^2 \\ &- \frac{e^{4A}}{4} \sum_i X_i^{-2} G_{\rho\sigma}^i G^{i\rho\sigma} + V_3 \\ &- \epsilon^{\mu\nu\rho} \frac{1}{4} |\epsilon_{ijk}| a_i B_\mu^j \wedge G_{\nu\rho}^k, \end{aligned} \quad (18)$$

where the final line corresponds to the topological Chern-Simons (CS) term and the new potential is

$$V_3 = \sum_i \left[4 \frac{e^{-4A}}{X_i} - \frac{1}{2} \frac{e^{-8A}}{X_i^2} a_i^2 \right] + 2\kappa e^{-6A}. \quad (19)$$

We can now dualise the gauge fields to bring the action to the canonical form of a non-linear sigma model coupled to gravity [18]. To do this, we redefine the field strengths

$$G^i = X_i^2 e^{-4A} * DY_i, \quad DY_i = dY_i - \frac{1}{2} |\epsilon_{ijk}| a_j B^k \quad (20)$$

and rewrite the fields $e^{W_i} = e^{2A} X_i^{-1}$. This rewriting has the added bonus that the scalars are then canonically normalised. In performing this action, the CS terms remain and one can check that varying the gauge fields leads to the duality relations (20).

The structure of $\mathcal{N} = 2$ supergravity is now manifest. In particular, one can see that the scalar manifold corresponds to the coset $[SU(1,1)/U(1)]^3$ where each factor is parametrised by a complex coordinate

$$z_i = e^{W_i} + iY_i. \quad (21)$$

This is in line with expectations, since in [20] the same coset appears when ungauged supergravity is reduced on an S^2 . However, one important distinction here is that the R-symmetry is gauged so $W = 0$. To make the Kähler structure of the scalar target space more explicit, we can introduce a Kähler potential

$$\mathcal{K} = - \sum_i \log(\Re z_i). \quad (22)$$

Now that we understand the scalar manifold, it is relatively easy to extract the T-tensor

$$T = \sum_i^3 \left[\frac{1}{2} e^{-W_i} - \frac{1}{4} e^{\kappa} \sum_i^3 a_i e^{W_i} \right], \quad (23)$$

and check that it reproduces the expected terms in the potential (19). The required gauging of the R-symmetry can also be verified from reducing the Killing spinor equations from five dimensions.

We can now minimise the potential with the supersymmetry condition $a_1 + a_2 + a_3 = -\kappa$ [15] leading to the supersymmetric AdS_3 vacuum presented in [1], where the non-zero scalars are

$$e^{2A} = \frac{a_1 X_2 + a_2 X_1}{2}, \quad (24)$$

$$\frac{X_1}{X_3} = \frac{a_1(a_2 + a_3 - a_1)}{a_3(a_1 + a_2 - a_3)}, \quad \frac{X_2}{X_3} = \frac{a_2(a_1 + a_3 - a_2)}{a_3(a_1 + a_2 - a_3)}.$$

This is also a critical point of T as expected for supersymmetric critical points.

In terms of T , the AdS_3 radius is now $\ell = 1/(2T)$. One can then determine the central charge by using the standard holographic prescription [21, 22]

$$c_R = \frac{3\ell}{2G^{(3)}}, \quad (25)$$

and recalling that $G^{(3)} = G^{(5)}/\text{vol}_\Sigma$ where $G^{(5)} = \pi/(2N^2)$ in the conventions of [2]. The result is

$$c_R = \frac{12a_1a_2a_3\eta_\Sigma N^2}{2(a_1a_2 + a_1a_3 + a_2a_3) - a_1^2 - a_2^2 - a_3^2}, \quad (26)$$

which is, as expected, in perfect agreement with [1, 2]. As an added bonus, one can also confirm that the exact superconformal R-symmetry (7) agrees with (3.4) of [2].

5. Summary

In this letter, we have proposed a natural three-dimensional supergravity description of c -extremization for SCFTs with $\mathcal{N} = (0, 2)$ supersymmetry. In light of the work of [11], it is not too surprising that the T -tensor is the function being extremised. From the gravity perspective, it is already well understood [23] that the holographic c -function should be inversely proportional to the real superpotential, and for certain three-dimensional flows, this is the case [24]. The task remains to identify the gauged supergravities corresponding to the wrapped M5-brane examples presented in [2]. We also hope to identify three-dimensional gauged supergravities which arise from dimensional reductions of more generic wrapped-brane geometries, such as those discussed in [25, 26, 27].

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References

- [1] F. Benini and N. Bobev, Phys. Rev. Lett. **110**, 061601 (2013) [arXiv:1211.4030 [hep-th]].
- [2] F. Benini and N. Bobev, arXiv:1302.4451 [hep-th].
- [3] K. A. Intriligator and B. Wecht, Nucl. Phys. B **667**, 183 (2003) [hep-th/0304128].
- [4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].
- [5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].
- [6] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].
- [7] D. Martelli, J. Sparks and S. -T. Yau, Commun. Math. Phys. **268**, 39 (2006) [hep-th/0503183].
- [8] D. Martelli, J. Sparks and S. -T. Yau, Commun. Math. Phys. **280**, 611 (2008) [hep-th/0603021].
- [9] A. Butti and A. Zaffaroni, JHEP **0511**, 019 (2005) [hep-th/0506232].
- [10] R. Eager, arXiv:1011.1809 [hep-th].
- [11] Y. Tachikawa, Nucl. Phys. B **733**, 188 (2006) [hep-th/0507057].
- [12] P. Szepietowski, JHEP **1212**, 018 (2012) [arXiv:1209.3025 [hep-th]].
- [13] I. Bah, C. Beem, N. Bobev and B. Wecht, Phys. Rev. D **85**, 121901 (2012) [arXiv:1112.5487 [hep-th]].
- [14] I. Bah, C. Beem, N. Bobev and B. Wecht, JHEP **1206**, 005 (2012) [arXiv:1203.0303 [hep-th]].
- [15] J. M. Maldacena and C. Nunez, Int. J. Mod. Phys. A **16**, 822 (2001) [hep-th/0007018].
- [16] F. Benini, Y. Tachikawa and B. Wecht, JHEP **1001**, 088 (2010) [arXiv:0909.1327 [hep-th]].
- [17] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Class. Quant. Grav. **21**, 4335 (2004) [hep-th/0402153].
- [18] B. de Wit, I. Herger and H. Samtleben, Nucl. Phys. B **671**, 175 (2003) [hep-th/0307006].
- [19] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta and C. N. Pope *et al.*, Nucl. Phys. B **558**, 96 (1999) [hep-th/9903214].
- [20] E. O. Colgain and H. Samtleben, JHEP **1102**, 031 (2011) [arXiv:1012.2145 [hep-th]].
- [21] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
- [22] M. Henningson and K. Skenderis, JHEP **9807**, 023 (1998) [hep-th/9806087].
- [23] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Adv. Theor. Math. Phys. **3**, 363 (1999) [hep-th/9904017].
- [24] M. Berg and H. Samtleben, JHEP **0205**, 006 (2002) [hep-th/0112154].
- [25] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos and D. Waldram, JHEP **0611**, 053 (2006) [hep-th/0605146].
- [26] P. Figueras, O. A. P. Mac Conamhna and E. O. Colgain, Phys. Rev. D **76**, 046007 (2007) [hep-th/0703275 [HEP-TH]].
- [27] J. P. Gauntlett and O. A. P. Mac Conamhna, Class. Quant. Grav. **24**, 6267 (2007) [arXiv:0707.3105 [hep-th]].